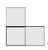
3 Topic: Principle of Mathematical Induction

The goal of this project is to practice the method of mathematical induction on several interesting (and sometimes important) examples.

Please note that the steps below in theorems are provided for your scratch work, which I do not expect to be submitted. The actual submission should only include a carefully stated and proved theorem with all necessary steps included either as lemmas or steps in the proof.

♢ **3.1.** Provide a proof by using the principle of mathematical induction that for any natural $n \geq 1$ a square grid with $2^n \times 2^n$ squares with one square removed can always be covered with L-shaped tiles that look like this: .

1. Check the base case $n = 1$.
2. Formulate the induction step and figure out how it can be used in the actual proof (this is the only nontrivial step in the whole proof). A few explicit examples can be useful ($n = 2, 3$).
3. Provide a proof for

   Theorem 3.1. *For any natural $n \geq 1$ an $2^n \times 2^n$ square grid with one square removed can be covered with L-shaped tiles.*

4. Using your proof formulate an algorithm how would you actually cover such a grid with L-shaped tiles step by step. Give an example for $n = 3$ (let's start with the square in row 3 and column 2 removed).

♢ **3.2.** Provide a detailed proof of the following *Newton's binomial theorem* (you have been reading about this theorem last week, but the textbook does not include a proof).

Theorem 3.2. *Let $a, b \in \mathbf{R}$ and $n \in \mathbf{N}$ be given. Then*

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k,$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

are the binomial coefficients (this is usually read "n choose k").

1. What is
$$\binom{5}{0}? \quad \binom{5}{5}? \quad \binom{5}{2}? \quad \binom{5}{3}?$$
2. Recognize that for $n = 2$ theorem gives familiar $(a + b)^2 = a^2 + 2ab + b^2$. Write down the expressions for
$$(a + b)^3, \quad (a + b)^4, \quad (a + b)^5$$
using the formula in the statement of the theorem.

3. Do you know the combinatorial meaning of the binomial coefficients? If not, google it. Come up with a combinatorial proof that $\binom{n}{k} = \binom{n}{n-k}$ (this, of course, trivially follows from the definition as well).

   Proof of the theorem:

4. Check the base of the induction for $n = 1$.
5. Formulate explicitly the induction step, or what you need to show.
6. Start with $(a+b)^{n+1}$ and use the induction step to manipulate your expression into two different sums. It could be beneficial to write separately the terms a^{n+1} and b^{n+1}. In one of the sums you will need to shift your summation index by one (it is done very similar to how you do substitutions in integrals; do not forget to change not only the index, but also the limits of summation).
7. Prove Pascal's identity
$$\binom{n}{k+1} + \binom{n}{k} = \binom{n+1}{k+1}.$$
8. (This is the only part from this problem I expect you to submit) Put together all the bits from the above and write the theorem statement along with its proof.

♢ **3.3** (All horses are the same color)**.** In 1961 in a satirical article by Joel E. Cohen, it was stated as a lemma, that "all horses are the same color." (This lemma allowed the author to subsquently "prove" that Alexander the Great did not exist and he had an infinite number of limbs (see for more detail Cohen, Joel E. (1961), "On the nature of mathematical proofs"). Here is a proof.

Proof. (by mathematical induction)

Base step $n = 1$. Clearly if the group of horses consists of just one horse then all the horses in this group are of the same color.

Induction step: Assume that my statement is true for given natural n. That is, I assume that in any group of horses of size n all the horses are of the same color. I need to show that it is also true for a group of $n + 1$ horses. I divide my $n + 1$ horses into two groups: first n and the last n. By my induction assumption in both groups the horses must be the same color. Moreover, since

$$\overbrace{1 \quad \underbrace{2 \quad 3 \quad \ldots \quad n-1 \quad n}_{n}}^{n} \quad \underbrace{}_{} n+1$$

these two groups intersect (see the diagram), the color of the first group must be the same as the color of the second group since they contain the same horses. Hence, the induction step is over, and by the principle of mathematical induction "all horses are the same color." ■

Where is the flaw in my reasonings?